

Path Integral in Holomorphic Representation without Gauge Fixation*

Sergei V. SHABANOV

*Department of Theoretical Physics,
University of Valencia,*

Moliner 50, Burjassot (Valencia), E-46100, Spain

and

Laboratory of Theoretical Physics, JINR, Dubna, Russia

Abstract

A method of path integral construction without gauge fixing in the holomorphic representation is proposed for finite-dimensional gauge models. This path integral determines a manifestly gauge-invariant kernel of the evolution operator.

1. It is well known that a gauge symmetry leads to constraints on dynamical variables in the theory [1]. Therefore, the evolution of unphysical degrees of freedom should be given when working with gauge theories, which implies gauge fixing. Alternatively, one can go over to gauge-invariant variables by means of an appropriate canonical transformation. In the latter case constraints turns into some of the new canonical momenta. Gauge-invariant variables are, in general, described by curvilinear coordinates, and their configuration space differs from the Euclidean space [2], [3]. In other words, a physical coordinate may take its value not on the whole real axis but only on its part (a halfline or a segment). Moreover physical degrees of freedom can have a phase space which differs from a plane [4], [5]. It leads to a modification of PI [5], and as a result, the quasi-classical description is changed [6].

According to the above remarks the following question can be raised: is there any way to construct PI which does not require elimination of unphysical degrees of freedom, and the evolution operator determined by such PI would be manifestly gauge-invariant? It is shown below that for finite-dimensional models with a gauge group (including the Yang-Mills quantum mechanics [7]) this question is not deprived of sense, and the recipe of finding PI that involves no gauge condition is proposed.

2. We shall explain the main idea of the note by a simple example where there is only one physical degree of freedom. The Lagrangian of the model is [4]

$$L = (\dot{x} - y_a T^a x)^2/2 - V(x^2) . \quad (1)$$

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Here an N -dimensional vector $x = (x_1, x_2, \dots, x_N)$ and $y_a (a = 1, 2, \dots, N)$ play the role of dynamical variables of the theory, T^a are $N \times N$ antisymmetric matrices which are generators of the group $SO(N)$, $[T^a, T^b] = f_{abc}T^c$, f_{abc} are structural constants of $SO(N)$, $(T^a x)_i = T_{ij}^a x_j$ and V is a potential. Lagrangian (1) remains invariable with respect to gauge transformations

$$x \rightarrow \Omega x \Omega^T, \quad y \rightarrow \Omega y \Omega^T - \Omega \partial_t \Omega^T, \quad y = y_a T^a, \quad (2)$$

where $\Omega = \exp \omega_a(t) T^a$, ω_a are arbitrary functions of time, Ω^T is the transposed matrix Ω .

Going over to the Hamiltonian formalism we find canonical momenta $\pi_a = \partial L / \partial \dot{y}_a = 0$ (primary constraints [1]) and $p = \partial L / \partial \dot{x} = \dot{x} - y_a T^a x$. The Hamiltonian is

$$H = p^2/2 + V(x^2) - y_a G^a, \quad (3)$$

where $G_a = \{\pi_a, H\} = p_i T_{ij}^a x_j = 0$ are secondary constraints ($\{, \}$ are Poisson brackets) which follow from the consistency condition $\dot{\pi}_a = 0$ [1]. All constraints are of the first class $\{G_a, G_b\} = f_{abc} G_c$, $\{G_a, H\} = -f_{abc} y_b G_c$. Thereby the quantization of the theory is carried out by the change of both the momenta and coordinates to operators with the commutation relations $[x_j, p_k] = i\delta_{jk}$, $[y_a, \pi_b] = i\delta_{ab}$, while the constraints select physical states [1]:

$$G_a |\psi_{ph}\rangle = 0, \quad \pi_a |\psi_{ph}\rangle = 0. \quad (4)$$

The second equality in (4) means that wave functions do not depend on y_a ; so below we shall not take these degrees of freedom into consideration. The first equation of (4) can easily be solved in the holomorphic representation. We define the operators [8] $\hat{a}_j = (x_j + ip_j)/\sqrt{2}$ and the representation $\hat{a}_j^+ \psi(a^*) = a_j^* \psi(a^*)$, $\hat{a}_j \psi(a^*) = \partial / \partial a_j^* \psi(a^*)$. The scalar product reads

$$\int d^N(a^*, a) (\psi_1(a^*))^* \psi_2(a^*) = \langle \psi_1 | \psi_2 \rangle, \quad (5)$$

where $d^N(a^*, a) = (2\pi i)^{-N} d^N a^* d^N a \exp(-a_j^* a_j)$. Any state in the holomorphic representation is decomposed over the basis $\langle a^* | n_1, \dots, n_N \rangle = \prod_{i=1}^N (a_i^*)^{n_i} / \sqrt{n_i!}$ here $n_i = 0, 1, \dots$. This basis is orthonormal with respect to the scalar product (5). The constraint operators become $G_a = T_{ij}^a \hat{a}_i^+ \hat{a}_j$. Note that here there is no operator ordering problem as T^a are antisymmetric matrices.

Clearly, the vacuum $\langle a^* | 0 \rangle = 1$ satisfies (4), so any physical state is determined by applying a function of the operators \hat{a}_j^+ which commutes with all the constraints G_a . Such a function can depend only on the operator $\hat{a}_j^+ \hat{a}_j^+$. Indeed, it must be invariant with respect to the $SO(N)$ -rotations of the vector \hat{a}_i^+ . The only independent invariant that can be built of this vector is its square. Consequently, we find the basis in the physical subspace

$$\langle a^* | n \rangle_{ph} = c_n (a_j^* a_j^*)^n, \quad n = 0, 1, \dots \quad (6)$$

The normalization factors c_n can be calculated from the equality $\langle n | n' \rangle_{ph} = \delta_{nn'}$ and (5):

$$c_n^{-2} = \left(\partial / \partial a_j^* \partial / \partial a_j^* \right)^n (a_j^* a_j^*)^n = 4^n n! \Gamma(n + N/2) / \Gamma(N/2). \quad (7)$$

Non-negative integers n_i ($i = 1, 2, \dots, N$) enumerate the total basis as the system contains N degrees of freedom, while the basis (6) is labelled only by one integer n , i.e., the system has only one physical degree of freedom. Note that from the gauge transformation law (2) follows that the absolute value of the position vector $r = (x^2)^{1/2} \geq 0$ plays the role of a physical variable. We remark that the phase space spanned by r and its canonical momentum p_r is a cone [4]. The fact that the physical configuration (or phase) space may not coincide with an Euclidean space is usually ignored in the PI construction for gauge theories. Incidentally, as has been shown in [5], it leads to a PI modification, and as a result, the quasiclassical description can be changed [6]. For a generic gauge system it is not always possible to establish the structure of the physical configuration (phase) space. This problem can be avoided if one uses the PI suggested below in which unphysical degrees of freedom are not eliminated explicitly.

Using the Feynman-Kac formula we write the evolution operator kernel in the physical subspace

$$U_t^{ph}(a^*, a) = \sum_E \psi_E^{ph}(a^*) (\psi_E^{ph}(a^*))^* e^{-iEt} , \quad (8)$$

where $\psi_E^{ph}(a^*)$ are eigenstates of the Hamiltonian (3) satisfying the Dirac condition (4). If in Eq.(8) we sum over all eigenstates of H , we get the kernel of the evolution operator $U_t(a^*, a)$ in the total Hilbert space. Our purpose is to establish a relation between U_t and U_t^{ph} without an explicit elimination of unphysical degrees of freedom by a gauge fixation.

Note that at $t = 0$, $U_t^{ph}(a^*, a) = Q(a^*, a)$ is the projector on the physical subspace, for the functions $\psi_E^{ph}(a^*)$ compose a complete orthonormal set. Note that H and the G_a commute and therefore the total Hilbert space can be decomposed into the orthogonal sum of physical and unphysical subspaces. According to this remark we deduce the equality

$$U_t^{ph}(a^*, a) = \int d^N(b^*, b) U_t(a^*, b) Q(b^*, a) , \quad (9)$$

i.e., the projection operator Q removes contributions of unphysical states to the evolution operator. There is a standard representation for the kernel $U_t(a^*, a)$ by PI [8]

$$U_t(a^*, a) = \int \prod_{\tau=0}^t \frac{d^N a^* d^N a}{(2\pi i)^N} \exp \left[\frac{1}{2} \left(a_j^*(t) a_j(t) + a_j^*(0) a_j(0) \right) \right] \exp iS , \quad (10)$$

where $a^*(t) = a^*$, $a(0) = a$ are the standard boundary conditions for PI in the holomorphic representation, $S = \int_0^t d\tau \left[i(a_j^* \dot{a}_j - \dot{a}_j^* a_j)/2 - H(a^*, a) \right]$ is the action of the system including unphysical degrees of freedom too; the kernel $H(a^*, a)$ is obtained from the operator H by replacing the operators \hat{a}_j^+ and \hat{a}_j by complex numbers a_j^* and a_j , respectively, after a rearrangement of all \hat{a}_j to the right from \hat{a}_j^+ .

Thus, the task is reduced to finding the kernel $Q(a^*, a)$. Since Q is the projector on a physical subspace and the vectors (6) form just another orthogonal basis in it,

we can use the latter to obtain the resolution of unity in the physical subspace

$$Q(a^*, a) = \sum_{n=0}^{\infty} c_n^2 (2\xi)^{2n} = \Gamma(N/2) \xi^{1-N/2} I_{N/2-1}(2\xi) , \quad (11)$$

where $\xi = 1/2(a_j^* a_j^* a_i a_i)^{1/2}$, I_ν is a modified Bessel function.

Formulas (9)-(11) solve the above task. The standard form for U_t^{ph} can also be given:

$$U_t^{ph}(a^*, a) = \int \prod_{\tau} \left(d^N(a^*, a) \mu(a^*, a) \right) \exp \Phi \exp i S_{ef} ; \quad (12)$$

here $\mu(a^*, a)$ is some measure in the total phase space of the system, S_{ef} is an effective action in it, Φ is a phase associated with a choice of boundary conditions (cf (10)). According to (8) the kernel of U_t^{ph} satisfies the equation $i\partial_t U_t^{ph}(a^*, a) = H(\hat{a}^+, \hat{a}) U_t^{ph}(a^*, a)$ with the initial condition $U_{t=0}^{ph} = Q$. Note that the kernel (10) satisfies the same equation but with the other initial condition: $U_{t=0}(a^*, a) = \exp \sum a_j^* a_j$. From this equation we obtain the infinitesimal kernel of U_ε^{ph} , $\varepsilon \rightarrow 0$,

$$U_\varepsilon^{ph}(a^*, a) = Q(a^*, a) \exp [-i\varepsilon H_{ef}(a^*, a)] + O(\varepsilon^2) , \quad (13)$$

$$H_{ef}(a^*, a) = Q^{-1}(a^*, a) H(a^*, \partial/\partial a^*) Q(a^*, a) . \quad (14)$$

Iterating the kernel (13) in accordance with the scalar product (5) we find the path integral representation of U_t^{ph} for a finite time in the form (12) where

$$\mu(a^*, a) = Q(a^*, a) ; \quad (15)$$

$$S = \int_0^t d\tau \left[\frac{1}{2iQ} \left(\dot{a}_j^* \frac{\partial}{\partial a_j^*} - \dot{a}_j \frac{\partial}{\partial a_j} \right) Q - H_{ef}(a^*, a) \right] ; \quad (16)$$

$$\Phi = a_j^*(t) a_j(t) - a_j^*(0) a_j(0) - \frac{1}{2} \ln \frac{Q(a^*(t), a(t))}{Q(a^*(0), a(0))} , \quad (17)$$

and $a^*(t) = a^*$, $a(0) = a$. Note, if there is no gauge symmetry, then $Q(a^*, a) = \exp a_j^* a_j$ and Eq.(12) turns into (10).

Thus, to avoid an explicit elimination of nonphysical variables in PI, there are two ways: either to use the projection formula (9) or to change both the measure and action according to formula (12), (14)-(17) in the ordinary PI over the total phase space. The main problem in both cases is to find the operator Q .

3. Now consider systems with several physical degrees of freedom. Let us find the operator Q for the Yang-Mills quantum mechanics [7] with the group SU(2). The model is obtained from Yang-Mills theory [9] by imposing the condition that all fields depend only on time, i.e., they are homogeneous in space. The Lagrangian is [10]

$$L = \text{Tr}(\dot{x} - yx)^T (\dot{x} - yx) / 2 - V(x) ; \quad (18)$$

here x is a real 3×3 matrix, y is an antisymmetric matrix. If in the Yang-Mills Lagrangian we identify potentials $A_i^a = A_i^a(t)$ with x_{ai} , where $i, a = 1, 2, 3$ enumerate spatial and isotopic coordinates, respectively, and $y_{ab} = -g\varepsilon_{abc} A_0^c$, g is a coupling constant, we get Lagrangian (18) in which $V = g^2/4[(\text{Tr} x^T x)^2 - \text{Tr}(x^T x)^2]$, however, our consideration does not depend on the potential form.

Lagrangian (18) is invariant with respect to gauge transformations of the form (2) where the vector x should be replaced by a matrix x and Ω is considered as an orthogonal 3×3 matrix. The Hamiltonian formalism for this model is also analogous to that of the model (1). The momentum canonical conjugated to y vanishes, so we shall not take this degree of freedom into consideration. The secondary constraints are generators of isotopic rotations of columns of a matrix x . Any real matrix x can be written in the polar representation $x = u\rho$, where u is an orthogonal matrix and ρ is a positive symmetrical matrix. Clearly, u contains only unphysical degrees of freedom (they can be eliminated by the gauge transformation $x \rightarrow u^T x$). If the PI is constructed only for physical variables ρ (their number is six because $\rho = \rho^T$), the problem of integration over positive definite matrices arises. It is not equivalent to integration over \mathbb{R}^6 [10]. Finally, it should be remarked that the physical phase space of the model differs from the Euclidean space [4], [5]. So it is convenient to use the above given recipe for the gauge-fixing-free PI.

Note that after going over to the holomorphic representation for each component of the matrix x_{ai} , all physical states should be gauge-invariant $\psi_{ph}(\Omega a^*) = \psi_{ph}(a^*)$, where $a_{aj}^* = (x_{aj} - ip_{aj})/\sqrt{2}$, p_{aj} are canonical momenta for x_{aj} . One can convince oneself that any vector $\psi_{ph}(a^*)$ must be a function of the gauge invariant matrix $(a^{*T} a^*)_{ij} = a_{ai}^* a_{aj}^*$ which describes six physical degrees of freedom in this model. So the orthonormal basis in the physical subspace has the form

$$\langle a^* | n \rangle = c(n_{ij}) [(a^{*T} a^*)_{ij}]^{n_{ij}}, \quad n_{ij} = 0, 1, \dots; \quad (19)$$

here $i > j$. The vectors (19) are normalized by the scalar product (5) where $N = 9$ is a total number of degrees of freedom and $-\text{Tr} a^{*T} a = -a_{ia}^* a_{ai}$ is to be placed in the measure in the exponential argument instead of $-a_j^* a_j$. The normalization factors $c(n_{ii})$ (no summation over i) are obtained from (7) by setting $n = n_{ii}$, $N = 3$, while to get $c(n_{ij})$, $i < j$, one should omit the factor 4^n in (7) and set $n = n_{ij}$, $N = 6$. Now we use again the resolution of unity in the physical subspace to find Q . A calculation similar to (11) yields

$$Q(a^*, a) = \pi^{3/2} \prod_{i=1}^3 \xi_{ii}^{-1/2} I_{1/2}(\xi_{ii}) \prod_{i < j=1}^3 \xi_{ij}^{-2} I_2(2\xi_{ij}), \quad (20)$$

where $\xi_{ij} = [(a^{*T} a^*)_{ij} (a^T a)_{ij}]^{1/2}$. Further by formula (14)-(17) we restore the physical (gauge-invariant) evolution operator (12) or we can apply (9).

4. In the conclusion we shall show the group method for calculating the operator Q in any gauge model with a finite number of degrees of freedom. Let the brackets \langle, \rangle mean the scalar product in a representation space of a compact gauge group G and T_g be a group element in this representation. Then

$$Q(a^*, a) = \mu_G^{-1} \int d\mu(g) \exp \langle a_i^*, T_g a_i \rangle; \quad (21)$$

here μ_G is a volume of the group space, $d\mu(g)$ is a right- and left-invariant Haar measure on G , the index i enumerates "particles" in a representation space, i.e., degrees of freedom are enumerated by i and the group index on which operators

T_g act. The operators T_g are assumed to be unitary with respect to the scalar product $\langle T_g a_i^*, T_g a_i \rangle = \langle a_i^*, T_g^+ T_g a_i \rangle = \langle a_i^*, a_i \rangle$, i.e., $T_g^+ = T_g^{-1}$. Now we verify easily that $Q(a^*, a) = Q(T_g a^*, T_g a)$. The latter follows from the unitarity of T_g and the invariance of the measure $d\mu(g_1 g g_2) = d\mu(g)$. It remains for us to prove the projective properties of Q . After simple calculations we get

$$\int d^N(b^*, b) Q(a^*, b) \psi(b^*) = \mu_G^{-1} \int d\mu(g) \psi(T_g^+ a^*) , \quad (22)$$

where N is a total number of degrees of freedom. To derive equality (22), we have used definition (21) and the change of integration variables $b_i^* \rightarrow b_i^* - T_g^+ a_i^*$ has been done. If $\psi(T_g a^*) = \psi(a^*)$, i.e., it is a physical state, Q acts as the unit operator because it is a projector on the physical subspace as follows from Eq.(22). The derivation of PI without gauge fixation in the Lagrange form will be given elsewhere.

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